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LETTER TO THE EDITOR

# Marginally inhomogeneous semi-infinite two-dimensional Ising models

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**Abstract.** The scaling dimensions of the semi-infinite two-dimensional Ising model with smoothly inhomogeneous couplings are determined, in general, by the eigenvalues of a Schrödinger-type equation. We obtain exact results for a marginal, two-parameter inhomogeneity that corresponds to a supersymmetric quantum mechanical potential. The results are compared with predictions of conformal invariance.

Two-dimensional Ising models with several types of marginal inhomogeneities in semi-infinite and bulk geometries have been studied, including smooth inhomogeneities that vary with distance from a free surface [1–9] or from an internal line [10, 11], and radially symmetric inhomogeneities [12, 13]. The non-universality of their local critical behaviours near the inhomogeneities was explained using a scaling argument [14]. In spite of the presence of inhomogeneities, the critical properties have been studied in terms of the conformal invariance. Iglói and others (see [6–8]) transformed the problem of calculating scaling dimensions into that of solving an ordinary differential equation. One can show (see below) that this procedure leads, in general, to a Schrödinger-type equation. Formulated in this way, the structure of solubility of the specific problem can be understood. (Here we consider only the semi-infinite geometry, in which the boundary condition is simpler to deal with than that in the bulk.) In this work we consider a marginally inhomogeneous semi-infinite two-dimensional Ising model (or a quantum Ising chain) with two free parameters, generalizing the work of [8].

Let us denote by $t(z)$ the local deviation of the temperature variable at the complex coordinate $z = x + iy$ from the criticality in inhomogeneous models in the continuum. Then $t(z)$ on the upper half-plane ($y > 0$) transforms [8] as

$$t(w) = |w'(z)|^{-y_t} t(z) \tag{1}$$

under the conformal transformation $w = w(z)$. In this work we will consider inhomogeneites of the form

$$t(z) = \frac{\alpha}{y^s} - \beta\left(\frac{x}{y|z|}\right)^s \tag{2}$$

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where $\alpha$ and $\beta$ are two free parameters. When $\beta = 0$, equation (2) reduces to that considered in [8]. With the scaling transformation $w = b^{-1}z$, we find

$$t(w) = \frac{\alpha'}{v^s} - \beta'\left(\frac{u}{v|w|}\right)^s \tag{3}$$

where $w = u + iv$ and

$$\alpha' = b^{y_t - s}\alpha \qquad \beta' = b^{y_t - s}\beta\,. \tag{4}$$

Hence the inhomogeneities are irrelevant, marginal, and relevant for $y_t < s$, $y_t = s$, and $y_t > s$, respectively [14]. Under the conformal mapping $w = (N/\pi)\ln z$ [15], we obtain in the strip $0 < v < N$

$$t(w) = \left(\frac{\pi}{N}\right)^{y_t} \exp[(y_t - s)\pi u/N]\left(\alpha \csc^s \frac{\pi v}{N} - \beta \cot^s \frac{\pi v}{N}\right). \tag{5}$$

In particular, the temperature variable of the marginally inhomogeneous Ising model ($y_t = s = 1$) takes the form

$$t(w) = \frac{\pi}{N}\left(\alpha \csc \frac{\pi v}{N} - \beta \cot \frac{\pi v}{N}\right) \tag{6}$$

and is homogeneous along the strip.

In the strip geometry, the problem is equivalent in the extreme anisotropic limit [16] to considering the inhomogeneous quantum Ising chain with free boundary condition and the Hamiltonian

$$H = -\tfrac{1}{2}\sum_{j=1}^{N-1} \lambda(j)\sigma_j^x\sigma_{j+1}^x - \tfrac{1}{2}\sum_{j=1}^{N}\sigma_j^z\,. \tag{7}$$

Here $1 - \lambda(j)$ is equal to $t(v = j)$, and $\sigma_j^x$, $\sigma_j^y$, and $\sigma_j^z$ are Pauli matrices at site $j$. The normalization is chosen so that the sound velocity is unity for the homogeneous Ising chain ($\lambda(j) = 1$) at criticality [17].

As is well known [18], the Hamiltonian (7) can be cast into a diagonal form

$$H = \sum_k \Lambda_k(a_k^\dagger a_k - \tfrac{1}{2}) \tag{8}$$

in fermion operators $a_k$ and $a_k^\dagger$ with the excitation energies $\Lambda_k$ given by the solution of an $N \times N$ eigenvalue problem. We are interested in the finite size scaling, where surface scaling dimensions $\Delta_k$ corresponding to one-fermion excitations are given [15] by

$$\Delta_k = \lim_{N\to\infty} \frac{N}{\pi}\Lambda_k\,. \tag{9}$$

Following the method in [6], we see that the difference equations satisfied by the eigenvectors become in the thermodynamic limit the first-order coupled differential equations

$$\begin{aligned} \frac{d\phi_k(x)}{dx} + W(x)\phi_k(x) &= -\Delta_k\psi_k(x) \\ -\frac{d\psi_k(x)}{dx} + W(x)\psi_k(x) &= -\Delta_k\phi_k(x) \end{aligned} \tag{10}$$

with the boundary condition $\phi_k(x=0) = \psi_k(x=\pi) = 0$. Here $W(x)$ is defined by

$$W(x) = \lim_{N\to\infty} \frac{N}{\pi}\left(1 - \lambda\left(\frac{N}{\pi}x\right)\right). \tag{11}$$

Eliminating $\psi_k(x)$ in (10) gives the equation

$$-\frac{\mathrm{d}^2\phi_k(x)}{\mathrm{d}x^2} + \{W(x)^2 - W'(x)\}\phi_k(x) = \Delta_k^2\phi_k(x) \tag{12}$$

for $\phi_k(x)$, with

$$\phi_k(x)|_{x=0} = 0 \qquad \phi_k'(x) + W(x)\phi_k(x)|_{x=\pi} = 0. \tag{13}$$

The equation for $\psi_k(x)$ can be obtained by negating $W(x)$ in (12) and (13). We note that the above derivation of (12) and (13) from (7) is applicable to any type of inhomogeneities described by $\lambda(j)$, or $W(x)$.

Equation (12) is just the Schrödinger equation in one dimension with energy eigenvalues $\Delta_k^2$. In this respect, the problem considered in [6] may be thought of as a free quantum particle in a three-dimensional sphere. The pair of equations (10) reminds us of supersymmetric quantum mechanics [19], where $W(x)$ is the supersymmetric potential and $V_\pm(x) = W(x)^2 \pm W'(x)$ supersymmetric partner potentials. Many of the potentials for which the Schrödinger equation is exactly soluble have a shape-invariance property, allowing the determination of the spectrum and wavefunctions by algebraic means [20]. The inhomogeneity given by $W(x) = \alpha \csc x - \beta \cot x$ (see (6) and (11)) is an example of a shape-invariant potential [19]:

$$V_+(x;\alpha,\beta) = V_-(x;\alpha,\beta+1) + (\beta+1)^2 - \beta^2. \tag{14}$$

But the boundary condition (13) for (12) is different from that usually considered in quantum mechanics, leading to different solutions for the present problems, and seems to prevent us from exploiting the shape-invariant property.

Following Burkhardt and Iglói [8], we obtain the solution to (12). There are four regions in ($\alpha$, $\beta$)-plane which exhibit different excitation spectra. For $\alpha - \beta < -\frac{1}{2}$ and $\alpha + \beta < -\frac{1}{2}$ (region I), we have

$$\begin{aligned} \Delta_k &= \sqrt{(-\alpha + 1/2 + k - 1)^2 - \beta^2} \qquad k = 1, 2, \ldots \\ \Delta_0 &= 0. \end{aligned} \tag{15}$$

On the other hand, for $\alpha - \beta < -\frac{1}{2}$ and $\alpha + \beta > -\frac{1}{2}$ (region II),

$$\Delta_k = \sqrt{(\beta + k)^2 - \beta^2} \qquad k = 0, 1, \ldots. \tag{16}$$

For $\alpha - \beta > -\frac{1}{2}$ and $\alpha + \beta > -\frac{1}{2}$ (region III), we have

$$\Delta_k = \sqrt{(\alpha + 1/2 + k)^2 - \beta^2} \qquad k = 0, 1, \ldots \tag{17}$$

while for $\alpha - \beta > -\frac{1}{2}$ and $\alpha + \beta < -\frac{1}{2}$ (region IV),

$$\Delta_k = \sqrt{(-\beta + k)^2 - \beta^2} \qquad k = 0, 1, \ldots . \tag{18}$$

Other scaling dimensions are given as sums of several $\Delta_k$.

The above results for the inhomogeneity given by (2) generalize those of Burkhardt and Iglói [8] ($\beta = 0$). In regions I and III, the spectra depend continuously on two parameters ($\alpha$ and $\beta$), which have not been considered before. Unlike the $\beta = 0$ case, $\Delta_k$ for general $\beta$ are not linear in $k$, which would have been expected if $\Delta_k$s and their sums form conformal towers. In general we do not have a conformal tower-like structure, but only asymptotically for large $k$. Similar phenomena have been observed in systems with inhomogeneities breaking the translational invariance along the free surface [6,7,12,13], as is the case in the present problem. In the appendix, the spectrum of the semi-infinite two-dimensional Ising model with a radially symmetric marginal inhomogeneity is analysed using (12). In regions II and IV, the spectrum is independent of $\alpha$, a manifestation of universality. In regions I, II, and IV ($\alpha + \beta < -\frac{1}{2}$ or $\alpha - \beta < -\frac{1}{2}$), *the lowest eigenvalue* $\Delta_0$ *vanishes in the* thermodynamic limit. Thus the system exhibits a spontaneous surface magnetization in these regions. This can be understood in view of [8], since the temperature inhomogeneities near the $x$ axis have the form $t(x + \mathrm{i}y) \simeq -2(\alpha \mp \beta)/y$, depending on the sign of $x$.

In summary, we observe that the scaling dimensions of the inhomogeneous Ising models in the finite-size scaling limit are related to the Schrödinger-type equation. Then using results for supersymmetric potentials we recognize which type of inhomogeneities lead to exactly sovable differential equations. We then obtain exact results for a marginal, two-parameter inhomogeneity that corresponds to a supersymmetric quantum potential, generalizing the work of [8]. Finally we note that the above analysis may also be applied to the smooth inhomogeneity in the bulk.

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## Appendix

Bariev and Peschel [12] considered the local magnetic surface exponent of the semi-infinite (and also bulk) two-dimensional Ising model with a radially symmetric marginal inhomogeneity. The general formalism developed above enables one to calculate the complete spectrum. With $W(x) = \alpha$, equation (12) becomes

$$\phi_k''(x) + (\Delta_k^2 - \alpha^2)\phi_k(x) = 0 . \tag{A1}$$

The boundary condition at $x = 0$ gives

$$\phi_k(x) = \sin\left(\sqrt{\Delta_k^2 - \alpha^2}\, x\right) \tag{A2}$$

while the boundary condition at $x = \pi$ requires

$$\phi_k'(x) + W(x)\phi_k(x)|_{x=\pi} = \sqrt{\Delta_k^2 - \alpha^2}\, \cos\left(\sqrt{\Delta_k^2 - \alpha^2}\, \pi\right) + \alpha \sin\left(\sqrt{\Delta_k^2 - \alpha^2}\, \pi\right) = 0 . \tag{A3}$$

For $k = 0$ this equation reproduces the expressions (18) of [12] for the local magnetic surface exponent $\beta_{l,1}(a)$ with the identification $a = -2\pi\alpha$. In particular, for $|\alpha| \ll 1$ we obtain

$$\Delta_k = k + \frac{1}{2} + \frac{\alpha}{\pi\left(k + \frac{1}{2}\right)} + O(\alpha^2) .$$

Hence this system does not show a conformal tower-like structure.

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